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A note on the role of the buoyancy layer in a rotating stratified fluid

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In the theory of steady stratified rotating fluid motions developed by Barcilon & Pedlosky (1967) (hereafter referred to as B & P) for flows within a circular, cylindrical container it was asserted that the innermost boundary layer on the vertical side wall was *absent to lowest order* when the side wall is thermally insulated. That is to say, the buoyancy layer is not required to close the vertical mass flux. This result has been disputed in a recent note by Harrington & Johnson (1969) (hereafter referred to as H & J). It is the purpose of this note to reiterate the earlier result of B & P and to show that H & J is in error. Further the result is placed on firmer ground by using the fundamental dynamical characteristics of the inviscid interior and viscous boundary layers and by avoiding the algebraic snares of manipulating Fourier series.

It was shown in B & P (and using the notation introduced there) that for $O(E^{\frac{2}{3}}) < \sigma S < O(1)$ the region of flow outside the horizontal Ekman layers was divided into (i) a geostrophic and hydrostatic inviscid interior, (ii) a relatively thick viscous, *hydrostatic* boundary layer, in which, however, the azimuthal velocity is geostrophic, and (iii) a thin buoyancy layer in which the fluid motion is not in hydrostatic balance. For $O(E^{\frac{2}{3}}) < \sigma S < O(E^{\frac{1}{2}})$ the hydrostatic layer consists of two parts, one with thickness scale $E^{\frac{1}{4}}$, the other with scale $(\sigma S)^{\frac{1}{2}}$ while for $O(E^{\frac{1}{2}}) < \sigma S < O(1)$ it has a single length scale $(\sigma S)^{\frac{1}{2}}$.

For the purpose of this note it is not necessary to separate these two hydrostatic components and they may be treated as a single layer (as was pointed out in §4 of B & P).

Satisfying the boundary condition on the azimuthal velocity at the cylinder wall ($r = a$), yields

$$v_I(a,z) + \hat{v}(a,z) = 0, \tag{1}$$

where v_I is the interior velocity and $\hat{v}$ is the correction field contributed by the hydrostatic layer. It is important to note that the buoyancy layer plays no role in this matching. The condition that the side wall be insulated implies that

$$\frac{\partial T_I}{\partial r}(a,z) + \frac{\partial \hat{T}}{\partial r}(a,z) + \frac{\partial \tilde{T}}{\partial r}(a,z) = 0, \tag{2}$$

where $\partial\tilde{T}/\partial r$ is the correction to the heat flux provided by the buoyancy layer. Since both v_I and $\hat{v}$ are in geostrophic and hydrostatic balance the vertical derivative of (1) yields

$$\frac{\partial T_I}{\partial r} + \frac{\partial \hat{T}}{\partial r} = 0, \tag{3}$$

which with (2) implies that

$$\frac{\partial \tilde{T}}{\partial r}(a, z) = 0. \tag{4}$$

This result in conjunction with the condition of vanishing vertical velocity on $r = a$ implies that the amplitude of the buoyancy layer is, to lowest order, zero. This may also be seen by direct calculation by multiplying the second equation in H & J by $\sin m\pi z$ and integrating over the interval $0 \leqslant z \leqslant 1$ to yield

$$\int_0^1 A(z) \sin m\pi z \, dz = 0 \tag{5}$$

for all m, where A is the amplitude of the vertical velocity in the buoyancy layer.

The derivation presented here fails only at the two points $z = 0$ and $z = 1$ where the Ekman compatibility condition requires that in the hydrostatic boundary layer (B & P)

$$\left.\begin{aligned} \tfrac{1}{2}\hat{v} &= \pm \frac{E^{\frac{1}{2}}}{\sigma S} \frac{\partial \hat{T}}{\partial r} \\ &= \pm \frac{E^{\frac{1}{2}}}{\sigma S} 2 \frac{\partial \hat{v}}{\partial z} \end{aligned}\right\} z = \tfrac{1}{2} \pm \tfrac{1}{2}. \tag{6}$$

Unless (and it does not occur)

$$v_I = \pm \frac{E^{\frac{1}{2}}}{\sigma S} 4 \frac{\partial v_I}{\partial z}, \quad z = \tfrac{1}{2} \pm \tfrac{1}{2}, \tag{7}$$

the balance (3) will not hold at the two end points and the resulting discrepancy will produce a contribution to the buoyancy layer at those *two points only*.

The contribution is (as can be seen from the aforementioned equation of H & J)

$$\left.\begin{aligned} A(z)/2^{\frac{1}{2}} &= -\tfrac{1}{2} v_T(a), & z &= 1, \\ &= \tfrac{1}{2} v_B(a), & z &= 0. \end{aligned}\right\} \tag{8}$$

This curious result merely states that the mass flux entering the side-wall boundary layer from the Ekman layer is initially partitioned between the hydrostatic and buoyancy layers but in a vertical distance much less than the vertical scale of the cylinder the entire vertical mass flux is contained in the hydrostatic layer, the buoyancy layer being absent for all $z \neq 0$ or 1 by virtue of the insulating condition at the side wall. The fact that $A(z)$ is not zero at $z = 0, 1$ is attributable to the discontinuous velocity boundary condition which occurs at the points where the rotating top and bottom adjoin the stationary side wall.

REFERENCES

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